# Exercises for 'Functional Analysis 2' [MATH-404]

(17/02/2025)

**Ex 1.1 (Some convex analysis)**
Let $X$ be a vector space. Recall that a subset $A \subset X$ is **convex** if, for all $x, y \in A$ and $\theta \in [0,1]$, it holds that $\theta x + (1-\theta)y \in A$. A function $f : A \to \mathbb{R}$ is convex if it satisfies $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ for all $x, y \in A$ and $\theta \in [0,1]$.

a) Show that if $(A_i)_{i\in I}$ is a family of convex subsets of $X$, then $A = \bigcap_{i\in I} A_i$ is also convex.

b) If $B \subset X$ is any set, we define its **convex hull** by

$$\operatorname{co}(B) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in B, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

Show that $\operatorname{co}(B)$ is the smallest convex set containing $B$.

c) Show that if $(f_i)_{i\in I} : A \to \mathbb{R}$ is a family of convex functions, then $f(x) := \sup_{i\in I} f_i(x)$ is convex, too.

**Solution 1.1:** a) Let $x, y \in A$ and $\theta \in [0,1]$. Then $x, y \in A_i$ for all $i \in I$ and by convexity of $A_i$ we have $\theta x + (1-\theta)y \in A_i$. Hence $\theta x + (1-\theta)y \in A$ which proves the convexity of $A$.

b) Clearly $B \subset \operatorname{co}(B)$ since $b = 1b$ for all $b \in B$. Next we prove the convexity. Fix $\theta \in [0,1]$. Let $x = \sum_{i=1}^{n} \lambda_i x_i \in \operatorname{co}(B)$ and $y = \sum_{i=1}^{m} \mu_i y_i \in \operatorname{co}(B)$, i.e, $x_i, y_i \in B$ and $\lambda_i, \mu \in [0,1]$ with

$$\sum_{i=1}^{n} \lambda_i = 1 = \sum_{i=1}^{m} \mu_i.$$

By a simple calculation we find that

$$\theta x + (1-\theta)y = \sum_{i=1}^{n} \theta \lambda_i x_i + \sum_{i=1}^{m} (1-\theta)\mu_i y_i.$$

Note that $\theta\lambda_i, (1-\theta)\mu_i \in [0,1]$ and

$$\sum_{i=1}^{n} \theta\lambda_i + \sum_{i=1}^{m} (1-\theta)\mu_i = \theta + (1-\theta) = 1.$$

Hence by definition $\theta x + (1-\theta)y \in \operatorname{co}(B)$ and consequently $\operatorname{co}(B)$ is convex. It remains to show that every convex set $S$ such that $B \subset S$ contains also $\operatorname{co}(B)$. Let $(x_i)_{i=1}^{n} \subset B$. By induction on $n$ we prove that $x = \sum_{i=1}^n \lambda_i x_i \in S$ for any $(\lambda_i)_{i=1}^n$ with $\lambda_i \in [0,1]$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_i = 1$. The case $n = 1$ is trivial since $x_1 \in B \subset S$. If $n \geq 2$, then without loss of generality $\lambda_n \neq 1$ and

$$\sum_{i=1}^n \lambda_i x_i = (1-\lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i + \lambda_n x_n$$

Since $\frac{\lambda_i}{1-\lambda_n} \in [0,1]$ and $\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} = 1$, the induction hypothesis yields that $\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i \in S$ and by convexity of $S$ it follows that $x \in S$.

c) Let $x, y \in A$ and $\theta \in [0,1]$. Then for all $i \in I$ we have

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y) \leq \theta f(x) + (1-\theta) f(y).$$

Taking the supremum in $i$ on the left hand side yields the claim.

**Ex 1.2 (Properties of absorbing and balanced sets)**
Let $X$ be a vector space. Recall that a subset $A \subset X$ is:

- **absorbing** if for all $x \in X$ there exists $\varepsilon > 0$ such that for all $|t| \leq \varepsilon$ we have $tx \in A$;
- **balanced** if $\lambda A \subset A$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

Prove the following:

a) If $(A_i)_{i \in I}$ is a family of absorbing sets, then $\bigcup_{i \in I} A_i$ is absorbing. In addition, show that if $I$ is finite, also $\bigcap_{i \in I} A_i$ is absorbing.

b) If $(A_i)_{i \in I}$ is a family of balanced sets, then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are balanced set.

c) Let $A \subset X$ be an arbitrary set. Show that $\bigcup_{|\lambda| \leq 1} \lambda A$ is the smallest balanced set containing $A$, where $\lambda \in \mathbb{R}$ (the set is called the **balanced hull**).

d) The convex hull of a balanced set is balanced.

**Solution 1.2:** a) Let $x \in X$ and $i_0 \in I$. Then there exists $\varepsilon > 0$ such that for all $|t| \leq \varepsilon$ we have $tx \in A_{i_0} \subset \bigcup_{i \in I} A_i$. Moreover, if $I$ is finite, then for all $i \in I$ let $\varepsilon_i > 0$ be such that for all $|t| \leq \varepsilon_i$ it holds that $tx \in A_i$. Set $\varepsilon = \min_{i \in I} \varepsilon_i > 0$. Then for all $|t| \leq \varepsilon$ it holds that $tx \in A_i$ for all $i \in I$. Hence $tx \in \bigcap_{i \in I} A_i$.

b) Let $\lambda \in \mathbb{R}$ be such that $|\lambda| \leq 1$. Then

$$\lambda \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} \lambda A_i \subset \bigcup_{i \in I} A_i.$$

For the intersection one has to be more careful since the equality $f(A \cap B) = f(A) \cap f(B)$ is not true in general. However, the needed inclusion $\subset$ remains valid (for the scalar multiplication even equality holds true), so that

$$\lambda \left( \bigcap_{i \in I} A_i \right) \subset \bigcap_{i \in I} \lambda A_i \subset \bigcap_{i \in I} A_i.$$

c) Set $B = \bigcup_{|\lambda| \leq 1} \lambda A$. Then $A = 1A \subset B$. Next, let us show that $B$ is balanced. Fix $\mu \in \mathbb{R}$

such that $|\mu| \leq 1$. For any $x \in B$ there exists $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ such that $x \in \lambda A$. Hence $\mu x \in (\mu\lambda)A \subset B$ since $|\lambda\mu| = |\lambda||\mu| \leq 1$. Finally, for any balanced set $C$ such that $A \subset C$ we infer from the definition of balancedness that $\lambda A \subset \lambda C \subset C$ for any $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. Thus $B \subset C$ which shows $B$ is the smallest balanced set containing $A$.

d) Let $A$ be a balanced set and denote by $\mathrm{co}(A)$ its convex hull. Then for any $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ and $x = \sum_{i=1}^{n} \lambda_i x_i \in \mathrm{co}(A)$ it holds that

$$\lambda x = \sum_{i=1}^{n} \lambda \lambda_i x_i = \sum_{i=1}^{n} \lambda_i \underbrace{(\lambda x_i)}_{\in A} \in \mathrm{co}(A).$$

**Ex 1.3 (On convex neighborhoods of the origin in TVS)**
Let $(X, \tau)$ be a topological vector space (TVS).

a) Show that the interior of a convex set is convex.

b) Show that every neighborhood of the origin is an absorbing set.

c) Show that every neighborhood of the origin contains a balanced open neighborhood of the origin.
**Hint**: Use the continuity of the scalar multiplication.

d) Show that every convex neighborhood of the origin contains an absorbing, balanced, convex, open neighborhood of the origin.
**Hint**: You might prove/use that in any TVS the convex hull of an open set is open.

**Solution 1.3:** a) Let $A \subset X$ be convex and $x, y \in \mathrm{int}(A)$. Fix $\theta \in [0,1]$. Then there exist open sets $U_x, U_y \in \tau$ such that $x \in U_x \subset A$ and $y \in U_y \subset A$. Set $U = ((U_x) - x) \cap (U_y - y)$ which is an open set containing the origin. We claim that $\theta x + (1-\theta)y + U \subset A$. Then by definition $\theta x + (1-\theta)y \in \mathrm{int}(A)$. For $u \in U$ it holds that

$$\theta x + (1-\theta)y + u = \theta(x+u) + (1-\theta)(y+u) \in \theta U_x + (1-\theta)U_y \subset A,$$

where we used the convexity of $A$ for the last inclusion.

b) Denote by $N$ a neighborhood of the origin and let $U \in \tau$ be such that $0 \in U \subset N$. Fix $x \in X$. Suppose by contradiction that there exists a $\mathbb{R}$-valued sequence $t_n \to 0$ such that $t_n x \notin U$ for all $n \in \mathbb{N}$. Then the sequence $x_n := t_n x$ cannot converge to 0, which contradicts the continuity of the scalar multiplication.

c) Let $N$ and $U$ be as in b). Then by the continuity of the scalar multiplication there exists an open set $V$ in $\mathbb{R} \times X$ such that $\lambda x \in U$ for all $(\lambda, x) \in V$. In particular, $(0,0) \in V$. By the definition of the product topology the set $V$ contains an open set of the form $B_\delta(0) \times V'$ with $\delta > 0$ and $0 \in V' \in \tau$. This implies that

$$\lambda x \in U \subset N \quad \forall \lambda \in B_\delta(0),\, x \in V'.$$

We set

$$U' = \bigcup_{\lambda \in B_\delta(0)} \lambda V' \subset U$$

Then $U'$ is balanced (the proof being almost the same as for 1.2 c)). Moreover by the continuity of the scalar multiplication (or rather the division) it follows that $\lambda V'$ is open for all $\lambda \in$

$B_\delta(0) \setminus \{0\}$. Since $0 \in V'$, we can take the above union equivalently over all $\lambda \in B_\delta(0) \setminus \{0\}$, so that it is the union of open sets. Thus $U'$ is also open and contains the origin.

d) Let $N \subset X$ be convex such that there exists $U \in \tau$ with $0 \in U \subset N$. Then $V := \text{int}(N)$ is open and convex by a). Moreover, $0 \in U \subset V$. By c) there exists an open set $V' \in \tau$ that is balanced and satisfies $0 \in V' \in V$. Consider then the convex hull $\text{co}(V')$ of $V'$, which is balanced by 1.2 d). Moreover it is convex and contains $V'$, so that it is a neighborhood of the origin. Hence it is absorbing by b). It remains to show that it is open. We prove the hint in order to conclude (recall that $V'$ is open). Let $x = \sum_{i=1}^n \lambda_i x_i \in \text{co}(V')$. Without loss of generality we assume that $\lambda_1 \neq 0$. Then the set

$$\lambda_1 V' + \sum_{i=2}^{n} \lambda_i x_i$$

is open since $X$ is a TVS. Moreover, it contains the point $x$ and by definition it is contained in the convex hull of $V'$. Hence $\text{co}(V')$ is open.